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Star-Products and Quasi-Quantum Groups

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The purpose of the present paper is to show that Drinfeld's theories of quasiquantum groups (quasitriangular quasi-Hopf algebras) and of quasi-Lie bialgebras can be developed in terms of the star products on a simple quasitriangular Poisson–Lie group.

1. INTRODUCTION

Quantum groups appeared first as quantum algebras, i.e., as one-parameter deformations of the universal enveloping algebras of complex simple Lie algebras, in the study of the algebraic aspects of quantum integrable systems by Kirillov and Reshetikhin (1988, 1990), Kulich and Skilyanin (1985), Kulich *et al.* (1981), and Faddeev (1982). Subsequently it was shown that these algebras are also deeply rooted in other areas, such as exactly soluble statistical models (Yang, 1967), factorizable *S*-matrix theory (Zamoldchikov *et al.*, 1979), conformal field theory (Alvarez-Caumé *et al.*, 1989; Moore and Seiberg, 1989), and the quantum Hall effect (Jellal, 1997). Mathematically, these algebras are Hopf algebras (Abe, 1980) which are noncocommutative.

The quantum algebras related to trigonometric solutions of the quantum Yang–Baxter equation were axiomatically introduced as quasitriangular Hopf algebras independently by Drinfeld (1983a, 1987) and Jimbo (1985, 1986). Other approaches to quantum groups have been developed by Faddeev *et al.* (1988), Manin (1988), Woronowicz (1989), and flato and Sternheimer (1994a,b). The last approach is based on the fact that there exists a simultitude between the star-product (Bayen *et al.*, 1978a,b) formulation of quantum mechanics and functional realizations of quantum groups.

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The notion of a star-product on a Poisson manifold introduced by Flato *et al.* (1975, 1976) has been extensively studied. The existence of a star-product has been studied by Vey (1975) and Neroslavsky and Vlassov (1979), who proved the existence of a star-product on a symplectic manifold with a vanishing third De Rham cohomology group, and by De wilde and Lecompte (1983) in the general case. From a geometrical point of view, Omori *et al.* (1991) and Fedesov (1994) also constructed star-products for arbitrary symplectic manifolds.

The notion of quasi-Hopf algebras was introduced by Drinfeld (1990, 1991) in connection with solutions of the Knizhnik–Zamolodchikov equations. They arise naturally in certain conformal field theories (Dijkgraaf and Witten, 1990; Pasquier *et al.*, 1990); the main feature is an invertible element Φ obeying a certain pentagon "cocycle" condition. The quasi-Hopf algebra is required to be associative as algebra, but coassociative only up conjugation by this Φ.

As in Moreno and Valero (1990, 1992, 1994), where the authors show that Drinfeld's theory of triangular quantum groups can be developed in terms of the invariant star-products on a triangular Poisson lie group (the Poisson structure is given by an r-matrix satisfying the classical Yang–Baxter equation) and prove some theorems given by Drinfeld (1983b) about solutions of the triangular quantum Yang–Baxter equations, and in contrast to previous work (Mansour, 1997), where an h-deformation of a Lie bialgebra as Lie algebra is given in terms of an invariant star-product on the corresponding triangular Poisson lie group, here I show that a star-product on a simple quasitriangular Poisson–Lie group (the r-matrix is a solution of the generalized Yang–Baxter equation) leads to a quasibialgebra structure on the corresponding Lie algebra and a quasitriangular quasi-Hopf algebra (quasi-quantum group) structure on the associated quantized enveloping algebra and that each family of equivalent star-products generates only one quasi-quantum group and only one quasi-Lie bialgebra.

1.1. Preliminaries

Assuming that K is a field of characteristic zero and recall the following (Drinfeld, 1990):

Definition 1. A quasibialgebra is a set $(H, \Delta, \varepsilon, \Phi)$ where H is an associative K-algebra with unit 1, Δ a homomorphism $H \to H \otimes H$, ε a homomorphism $H \to K$, and Φ is an invertible element of $H^{\otimes 3}$ and a 3-cocycle such that the following equalities hold:

$$(\mathrm{id}\otimes\Delta)\Delta(a) = \Phi^{-1}\cdot(\Delta\otimes id)\Delta(a)\cdot\Phi$$

$$(\Delta\otimes\mathrm{id}\otimes\mathrm{id})(\Phi)\cdot(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(\Phi)$$

$$= (\Phi\otimes 1)\cdot(\mathrm{id}\otimes\Delta\otimes\mathrm{id})(\Phi)\cdot(1\otimes\Phi)$$

$$(\varepsilon\otimes\mathrm{id})\Delta = (\mathrm{id}\otimes\varepsilon)\Delta = \mathrm{id}$$

$$(\mathrm{id}\otimes\varepsilon\otimes\mathrm{id})(\Phi) = 1$$

Definition 2. A quasi-Hopf algebra is a set $(H, \Delta, \varepsilon, \Phi, \alpha, \beta, S)$ where $(H, \Delta, \varepsilon, \Phi)$ is a quasibialgebra $\alpha, \beta \in H$, and S is an automorphism of H such that

$$\sum_i S(b_i)\alpha c_i = \varepsilon(a)\alpha, \qquad \sum_i b_i\beta S(c_i) = \varepsilon(a)\beta$$

$$\sum_i S(X_i)\alpha Y_i\beta S(Z_i) = \mathbf{1}, \qquad \sum_j P_j\beta S(Q_j)\alpha R_j = \mathbf{1}$$

where

$$\Delta(a) = \sum_i b_i\otimes c_i; \qquad \Phi = \sum_i X_i\otimes Y_i\otimes Z_i; \qquad \Phi^{-1} = \sum_i P_i\otimes Q_i\otimes R_i$$

Definition 3. A quasitriangular quasi-Hopf algebra is a set $(H, \Delta, \varepsilon, \Phi, \alpha, \beta, S, R)$ where $(H, \Delta, \varepsilon, \Phi, \alpha, \beta, S)$ is a quasi-Hopf algebra and R is an invertible element of $H\otimes H$ satisfying

$$\Delta^{\mathrm{op}} = R\Delta R^{-1}$$

$$(\Delta\otimes\mathrm{id})R = \Phi_{312}^{-1}R_{13}\Phi_{132}R_{23}\Phi_{123}^{-1}$$

$$(\mathrm{id}\otimes\Delta)R = \Phi_{231}R_{13}\Phi_{213}^{-1}R_{12}\Phi_{123}$$

Definition 4. A quasi-Lie bialgebra is a triple $(\mathbf{g}, \delta, \gamma)$ where $\mathbf{g}$ is a Lie algebra, δ is a 1-cocycle

$$\delta:\quad \mathbf{g}\to\Lambda^2(\mathbf{g})$$

i.e.,

$$\delta[x, y] = [x\otimes 1 + 1\otimes x, \delta(y)] - [y\otimes 1 + 1\otimes y, \delta(x)]$$

and $\gamma\in\Lambda^3(\mathbf{g})\subset\mathbf{g}\otimes\mathbf{g}\otimes\mathbf{g}$ satisfies the following equalities:

$$\tfrac{1}{2}\mathrm{Alt}(\delta\otimes\mathrm{id})\delta(x) = [x\otimes 1\otimes 1 + 1\otimes x\otimes 1 + 1\otimes 1\otimes x, \gamma]$$

$$\mathrm{Alt}(\delta\otimes\mathrm{id}\otimes\mathrm{id})(\gamma) = 0$$

where Alt stands for the alternation.

2. STAR-PRODUCT AND THE QUASI-QUANTUM GROUP STRUCTURE ON THE QUANTIZED ENVELOPING ALGEBRA

Let G be a simple Lie group, $\mathbf{g}$ its simple Lie algebra, and let there exist an $r \in \Lambda^2(\mathbf{g})$ which satisfies the generalized Yang–Baxter equation,

$$[r_{12}, r_{13}] + [r_{13}, r_{23}] + [r_{12}, r_{23}] = -[t_{13}, t_{23}] \tag{2.1}$$

where $r_{12} = r \otimes 1$, $r_{13} = \sigma_{23} r_{12}$, $r_{23} = 1 \otimes r$ (σ_{23} is the permutation of the second and third factors in the tensor product), and t is a symmetric and ad-invariant element of $\mathbf{g} \otimes \mathbf{g}$, i.e.,

$$t_{21} = t, \qquad [\Delta(x), t] = 0 \quad \text{for} \quad x \in \mathbf{g} \tag{2.2}$$

From the fact that $t \in \mathbf{g} \otimes \mathbf{g}$ we obtain that

$$(\Delta \otimes \mathrm{id})(t) = t_{13} + t_{23} \tag{2.3}$$

$$(\mathrm{id} \otimes \Delta)(t) = t_{12} + t_{13} \tag{2.4}$$

and from (2.2) we have

$$[t_{12}, t_{13} + t_{23}] = 0 \tag{2.5}$$

$$[t_{23}, t_{12} + t_{13}] = 0 \tag{2.6}$$

The Poisson–Lie structure on the Lie group G associated to the r-matrix,

$$r = r^{\mu\nu} X_\mu \otimes X_\nu / r^{\mu\nu} = -r^{\nu\mu}$$

is given by

$$\{\phi, \psi\}(g) = r^{\mu\nu}(X^l_{\mu/g}(\phi) \cdot X^l_{\nu/g}(\psi) - X^r_{\mu/g}(\phi) \cdot X^r_{\nu/g}(\psi))$$

where X^l_μ (X^r_μ) is the basis of left (right) invariant vector fields corresponding to $\{X_\mu\}$ (basis of $\mathbf{g}$) and is given by

$$X^l_{\mu/g} = X^l_\mu(g) = T_e L_g X_\mu, \qquad X^r_{\mu/g} = X^r_\mu(g) = T_e R_g X_\mu \qquad \forall \in gG$$

where L_g (R_g) is the left (right) translation of $\mathbf{G}$, and $T_e L_g$ ($T_e R_g$) is the tangent map of L_g (R_g) in e (element unit of $\mathbf{G}$).

The quasitriangular Lie bialgebra structure on the corresponding Lie algebra is given by the algebra 1-cocycle

$$\begin{aligned} \delta:\quad & \mathbf{g} \to \mathbf{g} \otimes \mathbf{g} \\ & x \to \delta(x) = (\mathrm{ad}_x \otimes 1 + 1 \otimes \mathrm{ad}_x) r \end{aligned} \tag{2.7}$$

where ad stands for the adjoint representation of $\mathbf{g}$ on $\mathbf{g}$.

We recall also that a star-product on a Poisson–Lie group G is defined as a bilinear map

$$C^\infty(G) \times C^\infty(G) \to C^\infty(G)[[h]]$$
$$(\varphi, \psi) \to \varphi * \psi = \sum_{k\geq 0} C_k(\varphi, \psi)h^k \tag{2.8}$$

where

$$C_0(\varphi, \psi) = \varphi \cdot \psi$$
$$C_1(\varphi, \psi) = \{\varphi, \psi\}$$

C_k are bidifferential operators on $C^\infty(G)$ null on the constants such that

$$\varphi * 1 = 1 * \varphi = 1 \tag{2.9}$$
$$(\varphi * \psi) * \phi = \varphi * (\psi * \phi) \tag{2.10}$$
$$\Delta(\varphi * \psi) = \Delta(\varphi) * \Delta(\psi) \tag{2.11}$$

where Δ is the usual coproduct on $C^\infty(G)$ defined by

$$\Delta(\varphi)(x, y) = \varphi(x \cdot y) \tag{2.12}$$

Takhtajan (1989) gives the desired star-product by the following expression:

$$\varphi * \psi = \mu[F^{-1}(x, y)^r F(x, y)^l (\varphi \otimes \psi)] \tag{2.13}$$

where μ is the usual multiplication on the algebra $C^\infty(G)$ and

$$F = 1 + \frac{h}{2} r + \sum_{k\geq 2} F_k h^k \in U(\mathbf{g})^{\otimes 2}[[h]] \tag{2.14}$$

[F^l (F^r) is the left (right) bidifferential operator corresponding to F] satisfies the following equation:

$$(\Delta_0 \otimes \mathrm{id})F \cdot (F \otimes 1) = \Phi \cdot (\mathrm{id} \otimes \Delta_0)F \cdot (1 \otimes F) \tag{2.15}$$

[Δ_0 is the usual coproduct of the enveloping algebra $U(\mathbf{g})$], where

$$\Phi = (\Delta_0 \otimes \mathrm{id})F \cdot (F \otimes 1) \cdot (1 \otimes F^{-1}) \cdot (\mathrm{id} \otimes \Delta_0)(F^{-1}) \tag{2.16}$$

is Ad_G-invariant.

In fact

$$\begin{aligned}(\varphi * \psi) * \phi &= \mu[F^{-1}(x, y)^r \cdot F(x, y)^l [\mu(F^{-1}(x, y)^r F(x, y)^l (\varphi \otimes \psi))] \otimes \phi] \\ &= \mu(\mu \otimes 1)[F^{-1}(x + y, z)^r F^{-1}(x, y)^r \\ &\quad \times F(x + y, z)^l F(x, y)^l (\varphi \otimes \psi \otimes \phi)] \\ &= \mu(\mu \otimes 1)[(F^{-1}(x, y) F^{-1}(x + y, z))^r \\ &\quad \times (F(x + y, z) F(x, y))^l (\varphi \otimes \psi \otimes \phi)]\end{aligned}$$

$$= \mu(\mu \otimes 1)[(\Phi^{-1}(x, y, z)^r \Phi(x, y, z)^l (F^{-1}(y, z) \times F^{-1}(x, y + z))^r (F(x, y + z)F(y, z))^l (\varphi \otimes \psi \otimes \phi)]$$
$$= \mu(1 \otimes \mu)[(\Phi^{-1}(x, y, z)^r \Phi(x, y, z)^l (F^{-1}(y, z) \times F^{-1}(x, y + z))^r (F(x, y + z)F(y, z))^l (\varphi \otimes \psi \otimes \phi)]$$

and

$$\varphi * (\psi * \phi) = \mu[F^{-1}(x, y)^r \cdot F(x, y)^l (\varphi \otimes \mu(F^{-1}(x, y)^r F(x, y)^l (\psi \otimes \phi))]$$
$$= \mu(1 \otimes \mu)[(F^{-1}(y, z)F^{-1}(x, y + z))^r \times (F(x, y + z)F(y, z))^l (\varphi \otimes \psi \otimes \phi)]$$

Then the associativity equation (2.10) for any $\varphi, \psi, \phi \in C^\infty(G)$ implies that

$$\Phi^{-1}(x, y, z)^r \cdot \Phi(x, y, z)^l = 1$$

i.e.,

$$T_e R_g \Phi^{-1} \cdot T_e L_g \Phi = 1$$
$$T_e R_g \Phi^{-1} \cdot T_e R_g (\mathrm{Ad}_g \Phi) = 1$$
$$T_e R_g (\Phi^{-1} \cdot \mathrm{Ad}_g \Phi) = 1 \qquad \text{for any } g \in G$$

which implies that

$$\mathrm{Ad}_g \Phi = \Phi$$

Precisely a star-product does not only define a deformation of $C^\infty(G)$, but also of the quotient algebra $\mathcal{F}_e^\infty(G)$ of $C^\infty(G)$ defined as the set of the C^∞-functions in a neighborhood containing the identity (e) of G modulo the following relation of equivalence:

$$\varphi \sim \psi \quad \text{if} \quad \langle X, \varphi - \psi \rangle = 0 \qquad \text{for any} \quad X \in U(\mathbf{g})$$

where

$$\langle X, \varphi \rangle = X(\varphi)(e)$$

The star-product is by definition compatible with this relation of equivalence, i.e.,

$$\text{if} \quad \varphi \sim \varphi' \quad \text{and} \quad \psi \sim \psi', \quad \text{then} \quad \varphi * \psi \sim \varphi' * \psi'$$

So the deformation we talk about is a deformation of $\mathcal{F}_e^\infty(G)$ as a bialgebra. This allows us to provide by the duality the deformed algebra ($D'(\{e\})[[h]]$, where $D'(\{e\})$ is the algebra of distributions on G with support at the unit element $e \in G$), or, thanks, to the theorem of L. Schwartz, which states that the enveloping algebra $U(\mathbf{g})$ is isomorphic to the subspace of distributions on G with support at the unit element $e \in G$, i.e.,

$$U(\mathbf{g}) \equiv D'(\{e\})$$

we deduce that a star-product provides a deformation of the enveloping algebra.

The quantized enveloping algebra $U(\mathbf{g})[[h]]$ is endowed with a structure of noncommutative Hopf algebra where the multiplication algebra is the ordinary convolution on $D'(\{e\})$ and the coproduct Δ_h is given by

$$\langle \Delta_h(X), \varphi \otimes \psi \rangle = \langle X, \varphi * \psi \rangle \qquad \text{for } X \in U(\mathbf{g}) \quad \text{and} \quad \varphi, \psi \in \mathcal{F}_e^\infty(G)$$

Explicitly

$$\begin{aligned}\langle X, \varphi * \psi \rangle &= \langle X, m((F^{-1})^r(F^l)(\varphi \otimes \psi)) \rangle \\ &= \langle \Delta_0(X), (F^{-1})^r(F^l)(\varphi \otimes \psi) \rangle\end{aligned}$$

Using the fact that

$$X_{/e}^{l(r)}(\varphi) = X_{/e}(\varphi) = X(\varphi)(e)$$

we obtain that

$$\langle X, \varphi * \psi \rangle = \mu((\Delta_0(X))^l(F^{-1})^r(F^l)(\varphi \otimes \psi))(e, e)$$

Then we must calculate first the quantity

$$I = ((\Delta_0(X))^l \cdot (F^{-1})^r \cdot F^l(\varphi \otimes \psi))(g, g)$$

For this we use the fact that for $X \in D'(\{e\})$ we have

$$X_{/g}^l(\varphi) = X^l(\varphi)(g) = \langle \delta_g *_c X, \varphi \rangle$$

$$X_{/g}^m(\varphi) = X^m(\varphi)(g) = \langle X *_c \delta_g, \varphi \rangle$$

where $*_c$ is the convolution product on $D'(\{e\})$ and δ_g is the Dirac distribution at $g \in G$. So,

$$\begin{aligned}I &= \langle (\delta_g \otimes \delta_g) *_c \Delta_0(X), (F^{-1})^r \cdot F^l(\varphi \otimes \psi) \rangle \\ &= \langle (\delta_g \otimes \delta_g) *_c \Delta_0(X), \langle F^{-1} *_c (\delta_g \otimes \delta_g), (\delta_g \otimes \delta_g) *_c F, \varphi \otimes \psi \rangle \rangle\end{aligned}$$

Next we use the following notation; for $X \in D'(\{e\})$, its dual (denoted $X^{\sim}$) $\in \mathcal{F}_e^\infty(G)$; then

$$\begin{aligned}I &= \langle F^{-1} *_c (\delta_g \otimes \delta_g), (\delta_g \otimes \delta_g) *_c F, \varphi \otimes \psi \rangle \rangle \cdot (\Delta_0(X))^{\sim}(g, g) \\ &= ((F^{-1})^{\sim} \cdot (\varphi \otimes \psi) \cdot (F)^{\sim} \cdot (\Delta_0(X))^{\sim})(g, g)\end{aligned}$$

and if we use the following property of the convolution product

$$(Y *_c X)^{\sim} = X^{\sim} \cdot Y^{\sim}$$

we have

$$\begin{aligned} I &= ((F^{-1})^{\sim} \cdot (\varphi \otimes \psi) \cdot (\Delta_0(X) *_c F)^{\sim})(g, g) \\ &= \langle (\delta_g \otimes \delta_g) *_c (\Delta_0(X) *_c F), (F^{-1})^{\sim} \cdot (\varphi \otimes \psi) \rangle \\ &= \langle (F^{-1}) *_c (\delta_g \otimes \delta_g) *_c (\Delta_0(X) *_c F), (\varphi \otimes \psi) \rangle \end{aligned}$$

Then we have

$$\begin{aligned} \langle X, \varphi * \psi \rangle &= \mu((\Delta_0(X))^l (F^{-1})^r (F^l)(\varphi \otimes \psi))(e, e) \\ &= \langle (F^{-1}) *_c (\delta_e \otimes \delta_e) *_c (\Delta_0(X) *_c F), (\varphi \otimes \psi) \rangle \\ &= \langle F^{-1} *_c \Delta_0(X) *_c F, (\varphi \otimes \psi) \rangle \end{aligned}$$

which implies that

$$\Delta_F(X) = F^{-1} *_c \Delta_0(X) *_c F \tag{2.17}$$

or the convolution product is the ordinary multiplication on the enveloping algebra $U(\mathbf{g})$, so (2.17) can be rewritten as

$$\Delta_F(X) = F^{-1} \cdot \Delta_0(X) \cdot F \tag{2.18}$$

Note that the star-product associativity (2.10) implies the coassociativity of the deformed coproduct Δ_F, i.e.,

$$(\Delta_F \otimes \mathrm{id})\Delta_F(X) = (\mathrm{id} \otimes \Delta_F)\Delta_F(X) \tag{2.19}$$

which implies that

$$(F^{-1}\Delta_0 F \otimes \mathrm{id})(F^{-1}\Delta_0(X)F) = (\mathrm{id} \otimes F^{-1}\Delta_0 F)(F^{-1}\Delta_0(X)F)$$

Again,

$$\begin{aligned} &(F^{-1} \otimes \mathbf{1}) \cdot (\Delta_0 \otimes \mathrm{id})F^{-1} \cdot (\Delta_0 \otimes \mathrm{id})\Delta_0(X) \cdot (\Delta_0 \otimes \mathrm{id})F \cdot (F \otimes \mathbf{1}) \\ &\quad = (\mathbf{1} \otimes F^{-1}) \cdot (\mathrm{id} \otimes \Delta_0)F^{-1} \cdot (\mathrm{id} \otimes \Delta_0)\Delta_0(X) \cdot (\mathrm{id} \otimes \Delta_0)F \cdot (\mathbf{1} \otimes F) \end{aligned}$$

so

$$\begin{aligned} (\Delta_0 \otimes \mathrm{id})\Delta_0(X) &= (\Delta_0 \otimes \mathrm{id})F \cdot (F \otimes \mathbf{1}) \cdot (\mathbf{1} \otimes F) \cdot (\mathrm{id} \otimes \Delta_0)F^{-1} \\ &\quad \times (\mathrm{id} \otimes \Delta_0)\Delta_0(X)(\mathrm{id} \otimes \Delta_0)F \cdot (\mathbf{1} \otimes F) \cdot (F^{-1} \otimes \mathbf{1}) \cdot (\Delta_0 \otimes \mathrm{id})F^{-1} \end{aligned}$$

Then

$$(\Delta_0 \otimes \mathrm{id})\Delta_0(X) = \Phi \cdot (\mathrm{id} \otimes \Delta_0)\Delta_0(X) \cdot \Phi^{-1}$$

or equivalently

$$(\mathrm{id} \otimes \Delta_0)\Delta_0(X) = \Phi^{-1} \cdot (\Delta_0 \otimes \mathrm{id})\Delta_0(X) \cdot \Phi \tag{2.20}$$

From (2.16) it is easily seen that Φ satisfies the pentagon equation

$$\begin{aligned}(\Delta_0 \otimes \mathrm{id} \otimes \mathrm{id})(\Phi) \cdot (\mathrm{id} \otimes \mathrm{id} \otimes \Delta_0)(\Phi) \\ = (\Phi \otimes 1) \cdot (\mathrm{id} \otimes \Delta_0 \otimes \mathrm{id})(\Phi) \cdot (1 \otimes \Phi)\end{aligned} \tag{2.21}$$

In fact

$$\begin{aligned}
&(\Delta_0 \otimes \mathrm{id} \otimes \mathrm{id})\Phi \cdot (\mathrm{id} \otimes \mathrm{id} \otimes \Delta_0)\Phi \\
&\quad = (\Delta_0 \otimes \mathrm{id} \otimes \mathrm{id})((\Delta_0 \otimes \mathrm{id})F \cdot (F \otimes 1) \cdot (1 \otimes F^{-1}) \cdot (\mathrm{id} \otimes \Delta_0)F^{-1}) \\
&\qquad \times (\mathrm{id} \otimes \mathrm{id} \otimes \Delta_0)((\Delta_0 \otimes \mathrm{id})F \cdot (F \otimes 1) \\
&\qquad \times (1 \otimes F^{-1}) \cdot (\mathrm{id} \otimes \Delta_0)(F^{-1})) \\
&\quad = (\Delta_0 \otimes \mathrm{id} \otimes \mathrm{id})((\Delta_0 \otimes \mathrm{id})F \cdot (F \otimes 1) \cdot (1 \otimes F^{-1})) \cdot (\Delta_0 \otimes \Delta_0)F^{-1} \\
&\qquad \times (\Delta_0 \otimes \Delta_0)F \cdot (\mathrm{id} \otimes \mathrm{id} \otimes \Delta_0) \\
&\qquad \times ((F \otimes 1) \cdot (1 \otimes F^{-1}) \cdot (\mathrm{id} \otimes \Delta_0)F^{-1}) \\
&\quad = (\Delta_0 \otimes \mathrm{id} \otimes \mathrm{id})((\Delta_0 \otimes \mathrm{id})F \cdot (F \otimes 1) \cdot (1 \otimes F^{-1}))(\mathrm{id} \otimes \mathrm{id} \otimes \Delta_0) \\
&\qquad \times ((F \otimes 1) \cdot (1 \otimes F^{-1}) \cdot (\mathrm{id} \otimes \Delta_0)F^{-1}) \\
&\quad = ((\Delta_0 \otimes \mathrm{id})\Delta_0 \otimes \mathrm{id})F \cdot ((\Delta_0 \otimes \mathrm{id})F \otimes 1) \cdot (F \otimes 1 \otimes 1) \\
&\qquad \times (1 \otimes F^{-1} \otimes 1) \cdot (1 \otimes F \otimes 1) \cdot (1 \otimes 1 \otimes F^{-1}) \\
&\qquad \times (1 \otimes (\mathrm{id} \otimes \Delta_0)F^{-1}) \cdot ((\mathrm{id} \otimes (\mathrm{id} \otimes \Delta_0)\Delta_0)F^{-1} \\
&\quad = ((\Delta_0 \otimes \mathrm{id})\Delta_0 \otimes \mathrm{id})F \cdot ((\Delta_0 \otimes \mathrm{id})F \cdot (F \otimes 1) \\
&\qquad \times (1 \otimes F^{-1}) \otimes 1) \cdot (1 \otimes (F \otimes 1) \cdot (1 \otimes F^{-1}) \\
&\qquad \times (\mathrm{id} \otimes \Delta_0)F^{-1}) \cdot ((\mathrm{id} \otimes (\mathrm{id} \otimes \Delta_0)\Delta_0)F^{-1} \\
&\quad = ((\Delta_0 \otimes \mathrm{id})\Delta_0 \otimes \mathrm{id})F \cdot (\Phi \cdot (\mathrm{id} \otimes \Delta_0)F \otimes 1) \\
&\qquad \times (1 \otimes (\mathrm{id} \otimes \Delta_0)F^{-1} \cdot \Phi) \cdot (\mathrm{id} \otimes (\mathrm{id} \otimes \Delta_0)\Delta_0)F^{-1} \\
&\quad = ((\Delta_0 \otimes \mathrm{id})\Delta_0 \otimes \mathrm{id})F \cdot (\Phi \otimes 1) \cdot ((\mathrm{id} \otimes \Delta_0)F \otimes 1) \\
&\qquad \times (1 \otimes (\mathrm{id} \otimes \Delta_0)F^{-1}) \cdot (1 \otimes \Phi) \cdot (\mathrm{id} \otimes (\mathrm{id} \otimes \Delta_0)\Delta_0)F^{-1} \\
&\quad = (\Phi \otimes 1) \cdot (\mathrm{id} \otimes \Delta_0 \otimes \mathrm{id})((\Delta_0 \otimes \mathrm{id})F \cdot (F \otimes 1) \\
&\qquad \times (1 \otimes F^{-1}) \cdot (\mathrm{id} \otimes \Delta_0)(F^{-1})) \cdot (1 \otimes \Phi) \\
&\quad = (\Phi \otimes 1) \cdot (\mathrm{id} \otimes \Delta_0 \otimes \mathrm{id})(\Phi) \cdot (1 \otimes \Phi)
\end{aligned}$$

The unitarity condition (2.9) implies that

$$(\varepsilon \otimes \mathrm{id})F = (\mathrm{id} \otimes \varepsilon)F = 1 \tag{2.22}$$

where ε is the usual counit of the enveloping algebra $U(\mathbf{g})$.

In polynomial notation we have

$$F(x, 0) = F(0, y) = 1$$

From (2.22) we can show that Φ satisfies the following equation:

$$(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})\Phi = 1 \tag{2.23}$$

In fact

$$\begin{aligned}(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})\Phi &= (\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})((\Delta_0 \otimes \mathrm{id})F \\ &\quad \times (F \otimes 1)\cdot(1 \otimes F^{-1})\cdot(\mathrm{id} \otimes \Delta_0)F^{-1}) \\ &= (\mathrm{id} \otimes \varepsilon)\Delta_0 \otimes \mathrm{id})F\cdot((\mathrm{id} \otimes \varepsilon)F \otimes 1) \\ &\quad \times ((1 \otimes (\varepsilon \otimes \mathrm{id})F^{-1})\cdot(\mathrm{id} \otimes (\varepsilon \otimes \mathrm{id})\Delta_0)F^{-1} \\ &= F\cdot F^{-1} \\ &= 1\end{aligned}$$

Using (2.21), we can show that Φ also satisfies

$$(\varepsilon \otimes \mathrm{id} \otimes \mathrm{id})\Phi = 1, \qquad (\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon)\Phi = 1 \tag{2.24}$$

We can show directly using polynomial notation

$$\begin{aligned}(\varepsilon \otimes \mathrm{id} \otimes \mathrm{id})\Phi(x, y, z) &= \Phi(0, y, z) \\ &= F(y, z)\cdot F(0, y)\cdot F^{-1}(y, z)\cdot F^{-1}(0, y + z) \\ &= F(y, z)\cdot F^{-1}(y, z) \\ &= 1 \\ (\mathrm{id} \otimes \mathrm{id}\ \varepsilon)\Phi(x, y, z) &= F(x + y, 0)\cdot F(x, y)\cdot F^{-1}(y, 0)\cdot F^{-1}(x, y) \\ &= F(x, y)\cdot F^{-1}(x, y) \\ &= 1\end{aligned}$$

Then from (2.18), (2.20), and (2.21) we deduce that $(U(\mathbf{g})[[h]], \Delta_0, \Phi)$ is a quasibialgebra and from equations (2.14), (2.16), (2.1) we can show that

$$\Phi(x, y, z) = 1 + \Phi_2(x, y, z)h^2 + \cdots \tag{2.25}$$

where

$$\text{Alt } \Phi_2 = -4[t_{13}, t_{23}] \tag{2.26}$$

In fact, for any element

$$F(x, y) = 1 + \sum_{k\geq 1} F_k(x, y)h^k \in U(\mathbf{g})^{\otimes 2}[[h]]$$

its inverse is given by

$$F^{-1}(x, y) = 1 + \sum_{k\geq 1} F_{\tilde{k}}(x, y)h^k$$

where

$$F_{\tilde{k}}(x, y) = -F_k(x, y) + \sum_{l,r\geq 1, l+r=k} F_l(x, y) \cdot F_{\tilde{r}}(x, y)$$

Then

$$\begin{aligned}\Phi_1(x, y, z) &= F_1(x, y) + F_1(x + y, z) - F_1(x, y + z) - F_1(y, z)\\ &= r_{12} + r_{13} + r_{23} - r_{13} - r_{12} - r_{23} = 0\end{aligned}$$

and a similar calculus (using the antisymmetry of r) shows that

$$\begin{aligned}&\Phi_2(x, y, z) + \Phi_2(z, x, y) + \Phi_2(y, z, x) - \Phi_2(x, z, y) - \Phi_2(y, x, z) - \Phi_1(z, y, x)\\ &\quad = 4([r_{12}, r_{13}] + [r_{13}, r_{23}] + [r_{12}, r_{23}])\end{aligned}$$

Finally the quasi-Lie bialgebra structure on $\mathbf{g}$ is given by the triplet $(\mathbf{g}, \delta, \gamma)$ where

$$\begin{aligned}\delta: \quad &\mathbf{g} \to \mathbf{g} \otimes \mathbf{g}\\ &x \to \delta(x) = 0\end{aligned} \tag{2.27}$$

and

$$\gamma = -1/4 \text{ Alt } \Phi_2 = [t_{13}, t_{23}] \tag{2.28}$$

in fact, from (2.1), (2.2) it can be seen that

$$[x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x, \gamma] = 0 \tag{2.29}$$

Now using the fact the antipode S_0 of the enveloping algebra $U(\mathbf{g})$ is an antiautomorphism of $U(\mathbf{g})$ satisfying

$$m(S_0 \otimes \text{id})\Delta_0(X) = m(\text{id} \otimes S_0)\Delta_0(X) = \varepsilon(X)\mathbf{1} \tag{2.30}$$

and that $F(F^{-1}) \in U(\mathbf{g})^{\otimes 2}[[h]]$ can be split as

$$F = \sum_k a_k \otimes b_k \qquad (F^{-1} = \sum_k c_k \otimes d_k) \tag{2.31}$$

and putting $u = \sum_k c_k S_0(d_k)$ as an invertible element of $U(\mathbf{g})$, then we can show that the antipode of the quantized enveloping algebra $(U(\mathbf{g})[[h]], \Delta_h, \varepsilon)$ is given by

$$S_F(X) = uS_0(X)u^{-1} \tag{2.32}$$

In fact, if we denote by $v = \sum_k S_0(a_k)b_k$, then applying the **k**-linear application

$$\Psi\colon\quad U(\mathbf{g})[[h]] \otimes U(\mathbf{g})[[h]] \otimes U(\mathbf{g})[[h]] \to U(\mathbf{g})[[h]]$$
$$(X_1 \otimes X_2 \otimes X_3) \to S_0(X_1)\cdot X_2 \cdot S_0(X_3)$$

to the equalities

$$(\Delta_0 \otimes \mathrm{id})F^{-1}\cdot\Phi = (F \otimes 1)\cdot(1 \otimes F^{-1})\cdot(\mathrm{id} \otimes \Delta_0)(F^{-1})$$
$$(\Delta_0 \otimes \mathrm{id})\Delta_0(X)\Phi = \Phi(\mathrm{id} \otimes \Delta_0)\Delta_0(X)$$

we obtain that

$$vu = \alpha \tag{2.33}$$

$$\alpha X = X\alpha \tag{2.34}$$

for any $X \in U(\mathbf{g})[[h]]$, where $\alpha = \Psi(\Phi) = \sum_i S_0(X_i)\cdot Y_i \cdot S_0(Z_i)$ for Φ given by

$$\Phi = \sum_i X_i \otimes Y_i \otimes Z_i \tag{2.35}$$

Similarly applying the **k**-linear application

$$\Psi'\colon\quad U(\mathbf{g})[[h]] \otimes U(\mathbf{g})[[h]] \otimes U(\mathbf{g})[[h]] \to U(\mathbf{g})[[h]]$$
$$(X_1 \otimes X_2 \otimes X_3) \to X_1 \cdot S_0(X_2)\cdot X_3$$

to the equalities

$$(\mathrm{id} \otimes \Delta_0)(F^{-1})\cdot\Phi^{-1} = (1 \otimes F)\cdot(F^{-1} \otimes 1)\cdot(\Delta_0 \otimes \mathrm{id})F^{-1}$$
$$\Phi^{-1}\cdot(\Delta_0 \otimes \mathrm{id})\Delta_0(X) = (\mathrm{id} \otimes \Delta_0)\Delta_0(X)\cdot\Phi^{-1}$$

we obtain that

$$uv = \beta \tag{2.36}$$

$$\beta X = X\beta \tag{2.37}$$

for any $X \in U(\mathbf{g})[[h]]$, where $\beta = \Psi'(\Phi^{-1}) = \sum_i P_i \cdot S_0(Q_i)\cdot R_i$, for Φ^{-1} given by

$$\Phi^{-1} = \sum_i P_i \otimes Q_i \otimes R_i \tag{2.38}$$

It is easily seen that α and β are invertible element in $U(\mathbf{g})[[h]]$, and using equations (2.33), (2.34), (2.36), (2.37), we remark that α and β are central elements in $U(\mathbf{g})[[h]]$, and

$$\alpha = \beta, \qquad u^{-1} = \alpha^{-1}v = v\alpha^{-1}$$

It remains to show that S_F defines an antipode on $U(\mathbf{g})[[h]]$, i.e.,

$$m(S_F \otimes \mathrm{id})\Delta_F(X) = m(\mathrm{id} \otimes S_F)\Delta_F(X) = \varepsilon(X)\mathbf{1} \tag{2.39}$$

where m is the usual multiplication on the enveloping algebra.

In fact

$$\begin{aligned} m(S_F \otimes \mathrm{id})\Delta_F(X) &= m(uS_0u^{-1} \otimes \mathrm{id})(F^{-1} \cdot \Delta_0(X) \cdot F) \\ &= \sum_{i,j,k} uS_0(a_i)S_0(X'_k)S_0(c_c)u^{-1}d_j X''_k b_i \\ &= \alpha^{-1} \sum_{i,j,k} uS_0(a_i)S_0(X'_k)S_0(c_j)vd_j X''_k b_i \end{aligned}$$

with $\Delta_0(X) = \sum_k X'_k \otimes X''_k$.

Owing the fact that

$$\sum_j S_0(c_j)vd_j = m(S_0 \otimes \mathrm{id})(F \cdot F^{-1}) = \mathbf{1}$$

and

$$\sum_k S_0(X'_k) \cdot X''_k = \varepsilon(X)\mathbf{1}$$

we have

$$\begin{aligned} m(S_F \otimes \mathrm{id})\Delta_F(X) &= \alpha^{-1} \sum_i uS_0(a_i)b_i\varepsilon(X) \\ &= \alpha^{-1}uv\varepsilon(X) \\ &= \varepsilon(X)\mathbf{1} \end{aligned} \tag{2.40}$$

Similarly, we can prove that

$$m(\mathrm{id} \otimes S_F)\Delta_F(X) = \varepsilon(X)\mathbf{1} \tag{2.41}$$

Finally, from the equalities

$$\sum_i S_0(X_i) \cdot Y_i \cdot S_0(Z_i) = \alpha = \sum_i P_i \cdot S_0(Q_i) \cdot R_i$$

$$\sum_k S_0(X'_k) \cdot X''_k = \varepsilon(X)\mathbf{1} = \sum_k X'_k \cdot S_0(X''_k)$$

we deduce that the quasi-Hopf algebra structure on the quasibialgebra $(U(\mathbf{g})[[h]], \Delta_0, \varepsilon, \Phi)$ is given by $(\alpha = \beta = 1, S_0)$.

Drinfeld (1987) showed that the quasitriangular structure on the quantized enveloping algebra $(U(\mathbf{g})[[h]], \Delta_F, \varepsilon, S_F)$ is given by the R-matrix,

$$R_F = F_{21}^{-1} e^{ht/2} F \tag{2.42}$$

which satisfies

$$(\Delta_F \otimes \mathrm{id}) R_F = (R_F)_{13}(R_F)_{23} \tag{2.43a}$$

$$(\mathrm{id} \otimes \Delta_F) R_F = (R_F)_{13}(R_F)_{12} \tag{2.43b}$$

and the quantum Yang–Baxter equation

$$(R_F)_{12}(R_F)_{13}(R_F)_{23} = (R_F)_{23}(R_F)_{13}(R_F)_{12} \tag{2.44}$$

where t is the symmetric element (2.2).

Then from equation (2.43a) rewritten in polynomial notation as

$$(\Delta_F \otimes \mathrm{id}) R_F(x, y) = R_F(x, z) R_F(y, z)$$

we obtain

$$\begin{aligned}&(F^{-1}\Delta_0 F \otimes \mathrm{id})(F^{-1}(y, x) e^{(ht/2)(x,y)} F(x, y)) \\ &\quad = F^{-1}(z, x) e^{(ht/2)(x,z)} F(x, z) F^{-1}(z, y) e^{(ht/2)(y,z)} F(y, z) F^{-1}(x, y)\end{aligned}$$

Then

$$\begin{aligned}&F^{-1}(z, x + y)(\Delta_0 \otimes \mathrm{id}) e^{(ht/2)(x,y)} F(x + y, z) F(x, y) \\ &\quad = F^{-1}(z, x) e^{(ht/2)(x,z)} F(x, z) F^{-1}(z, y) e^{(ht/2)(y,z)} F(y, z)\end{aligned}$$

which implies that

$$\begin{aligned}(\Delta_0 \otimes \mathrm{id}) e^{(ht/2)(x,y)} = {} & F(z, x + y) F(x, y) F^{-1}(z, x) e^{(ht/2)(x,z)} F(x, z) \\ & \times F^{-1}(z, y) e^{(ht/2)(y,z)} F(y, z) F^{-1}(x, y) F^{-1}(x + y, z)\end{aligned}$$

Finally, we obtain

$$\begin{aligned}(\Delta_0 \otimes \mathrm{id}) e^{(ht/2)(x,y)} = {} & F(z, x + y) F(x, y) F^{-1}(z, x) \\ & \times F^{-1}(z + x, y) e^{(ht/2)(x,z)} F(x + z, y) F(x, z) \\ & \times F^{-1}(z, y) F^{-1}(z + y, x) e^{(ht/2)(y,z)} \\ & \times F(y + z, x) F(y, z) F^{-1}(x, y) F^{-1}(x + y, z)\end{aligned} \tag{2.45}$$

and using (2.16), we obtain

$$(\Delta_0 \otimes \mathrm{id})e^{(ht/2)(x,y)} = \Phi^{-1}(z, x, y)e^{(ht/2)(x,z)}\Phi(x, z, y)e^{(ht/2)(y,z)}\Phi^{-1}(x, y, z) \tag{2.46}$$

Similarly from (2.43b) we show that

$$(\mathrm{id} \otimes \Delta_0)e^{(ht/2)(x,y)} = \Phi(y, z, x)e^{(ht/2)(x,z)}\Phi^{-1}(y, x, z)e^{(ht/2)(x,y)}\Phi(x, y, z) \tag{2.47}$$

Note that also

$$\begin{aligned} \Delta_F^{\mathrm{op}} &= \sigma(F^{-1})\Delta_0\sigma(F) \\ &= \sigma(F^{-1})e^{ht/2}\Delta_0 e^{-ht/2}\sigma(F) \\ &= \sigma(F^{-1})e^{ht/2}F\Delta_F F^{-1}e^{-ht/2}\sigma(F) \\ &= R_F\Delta_F R_F^{-1} \end{aligned} \tag{2.48}$$

In order to show that $R = e^{ht/2}$ satisfies the quasi-quantum Yang–Baxter equation, let us note that by hypothesis we have

$$F(x + y, z)F(x, y) = \Phi(x, y, z)F(x, y + z)\cdot F(y, z)$$

We first remark that a similar relation holds for any permutation of (x, y, z).

Clearly we have

$$\begin{aligned} &e^{-(ht/2)(x,y)}F(x + y, z)F(y, x)R_F(x, y) \\ &\quad = \Phi(x, y, z)e^{-(ht/2)(y,z)}F(x, y + z)\cdot F(z, y)R_F(y, z) \end{aligned}$$

and from the above remark

$$\begin{aligned} &e^{-(ht/2)(x,y)}\Phi(y, x, z)e^{-(ht/2)(x,z)}F(y, x + z)F(z, x)R_F(x, z)R_F(x, y) \\ &\quad = \Phi(x, y, z)e^{-(ht/2)(y,z)}\Phi^{-1}(x, z, y)e^{-(ht/2)(x,z)}F(x + z, y) \\ &\quad\quad \times F(z, x)R_F(x, z)R_F(y, z) \end{aligned}$$

Again

$$\begin{aligned} &e^{-(ht/2)(x,y)}\Phi(y, x, z)e^{-(ht/2)(x,z)}\Phi^{-1}(y, x, z)F(y + z, x)F(y, z)R_F(x, x)R_F(x, y) \\ &\quad = \Phi(x, y, z)e^{-(ht/2)(y,z)}\Phi^{-1}(x, z, y)e^{-(ht/2)(x,z)}\Phi(z, x, y)F(z, x + y) \\ &\quad\quad \times F(x, y)R_F(x, z)R_F(y, z) \end{aligned}$$

and in the same way

$$\begin{aligned} &e^{-(ht/2)(x,y)}\Phi(y, x, z)e^{-(ht/2)(x,z)}\Phi^{-1}(y, x, z)e^{-(ht/2)(y,z)}F(z + y, x) \\ &\quad \times F(z, y)R_F(y, z)R_R(x, z)R_F(x, y) \end{aligned}$$

$$= \Phi(x, y, z)e^{-(ht/2)(y,z)}\Phi^{-1}(x, z, y)e^{-(ht/2)(x,z)}\Phi(z, x, y)e^{-(ht/2)(x,y)}F(z, x + y) \times F(y, x)R_F(x, y)R_F(x, z)R_F(y, z)$$

Then

$$\begin{aligned} &e^{-(ht/2)(x,y)}\Phi(y, x, z)e^{-(ht/t)(x,z)}\Phi^{-1}(y, x, z)e^{-(ht/2)(y,z)}\Phi(z, y, x) \\ &\quad\times F(z, x + y)F(y, x)R_F(y, z)R_F(x, z)R_F(x, y) \\ &= \Phi(x, y, z)e^{-(ht/2)(y,z)}\Phi^{-1}(x, z, y)e^{-(ht/2)(x,z)}\Phi(z, x, y)e^{-(ht/2)(x,y)} \\ &\quad\times F(z, x + \mathrm{y})F(y, x)R_F(x, y)R_F(x, z)R_F(y, z) \end{aligned}$$

By using (2.44), we obtain that

$$\begin{aligned} &e^{-(ht/2)(x,y)}\Phi(y, x, z)e^{-(ht/2)(x,z)}\Phi^{-1}(y, x, z)e^{-(ht/2)(y,z)}\Phi(z, y, x) \\ &= \Phi(x, y, z)e^{-(ht/2)(y,z)}\Phi^{-1}(x, z, y)e^{-(ht/2)(x,z)}\Phi(z, x, y)e^{-(ht/2)(x,y)} \end{aligned}$$

which can be rewritten in compact form as

$$\Phi_{123}\, R_{23}^{-1}\Phi_{132}^{-1}\, R_{13}^{-1}\Phi_{132}\, R_{12}^{-1} = R_{12}^{-1}\Phi_{213}\, R_{13}^{-1}\Phi_{213}^{-1}\, R_{23}^{-1}\Phi_{321}$$

which implies that R satisfies the quasi-quantum Yang–Baxter equation

$$R_{12}\Phi_{312}^{-1}\, R_{13}\Phi_{132}\, R_{23}\Phi_{123}^{-1} = \Phi_{231}^{-1}\, R_{23}\Phi_{213}\, R_{13}\Phi_{213}^{-1}\, R_{12} \tag{2.49}$$

Then we deduce from (2.46), (2.47), (2.49) that $R = e^{ht/2}$ defines a quasitriangular quasi-Hopf algebra on $(U(\mathbf{g})[[h]], \Delta_0, \varepsilon, \Phi)$.

Remarks. (i) If we change the group multiplication into the opposite one, then the corresponding quasi-Hopf algebra is given by $(U(\mathbf{g})[[h]], \Delta_0, \varepsilon, \Phi', \alpha', \beta', S_0')$, where $U(\mathbf{g})[[h]]$ is the quantized enveloping algebra with the opposite multiplication, $\Phi' = \Phi^{-1}$, $S_0' = S_0^{-1}$, $\alpha' = S_0^{-1}(1) = 1$, $\beta' = S_0^{-1}(1) = 1$.

(ii) If the star-product is defined as the deformation of the opposite multiplication algebra of the C^∞-function algebra on G, then the resulting quasi-Hopf algebra is $(U(\mathbf{g})[[h]], \Delta_0^{op}, \varepsilon, \Phi', \alpha', \beta', S_0')$, where Δ_0^{op} is the opposite comultiplication, $\Phi' = \Phi_{321}^{-1}$, $S_0' = S_0^{-1}$, $\alpha' = S_0^{-1}(1) = 1$, $\beta' = S_0^{-1}(1) = 1$.

So, using the fact that the multiplication algebra of the C^∞-function algebra is commutative and that the coproduct Δ_0 satisfies $\Delta_0 = \Delta_0^{op}$, then the two quasibialgebra $(U(\mathbf{g})[[h]], \Delta_0^{op}, \varepsilon, \Phi)$ and $(U(\mathbf{g})[[h]], \Delta_0^{op}, \varepsilon, \Phi_{321}^{-1})$ must be the same, which implies that $\Phi_{321}^{-1} = \Phi$.

Let us suppose now that we have two star-products F and F' which generate two quasitriangular quasi-Hopf algebras $(U(\mathbf{g})[[h]], \Delta_0, \varepsilon, S_0, \Phi, e^{ht/2})$ and $(U(\mathbf{g})[[h]], \Delta_0, \varepsilon, S_0, \Phi', e^{ht/2})$, where

$$\Phi(x, y, z) = F(x+y, z)F(x, y)F^{-1}(y, z)F^{-1}(x, y+z) \tag{2.50}$$

$$\Phi'(x, y, z) = F'(x+y, z)F'(x, y)(F')^{-1}(y, z)(F')^{-1}(x, y+z) \tag{2.51}$$

If the star-products F and F' are equivalent, i.e., there exists an element $E(x) \in u(\mathbf{g})[[h]]$ such that

$$F'(x, y) = E^{-1}(x+y)F(x, y)E(x)E(y) \tag{2.52}$$

then

$$\begin{aligned}\Phi'(x, y, z) &= E^{-1}(x+y+z)F(x+y, z)E(x+y)E(z)E^{-1}(x+y)\\ &\quad\times F(x, y)E(x)E(y)E^{-1}(z)E^{-1}(y)F^{-1}(y, z)E(y+z)\\ &\quad\times E^{-1}(y+z)E^{-1}(x)F^{-1}(x, y+z)E(x+y+z)\\ &= E^{-1}(x+y+z)F(x+y, z)F(x, y)F^{-1}(y, z)\\ &\quad\times F^{-1}(x, y+z)E(x+y+z)\\ &= E^{-1}(x+y+z)\Phi(x, y, z)E(x+y+z)\\ &= \Phi(x, y, z)\end{aligned} \tag{2.53}$$

Then the two equivalent star-products generate only one quasitriangular quasi-Hopf algebra and obviously one quasi-Lie bialgebra.

If the star-products F and F' are not equivalent, then they generate two different quasi-Hopf algebras $(U(\mathbf{g})[[h]], \Delta, \varepsilon, \Phi)$ and $(U(\mathbf{g})[[h]], \Delta, \varepsilon, \Phi')$, which are isomorphic thanks to the following theorem.

Theorem (Drinfeld, 1990). Assume we have $A = (U(\mathbf{g})[[h]], \Delta, \varepsilon, \Phi, e^{ht/2})$ and $A' = (U(\mathbf{g})[[h]], \Delta, \varepsilon, \Phi', e^{ht/2})$, which are quantum enveloping algebras for the same finite-dimensional semisimple Lie algebra $\mathbf{g}$; then there exists a gauge transformation F in the algebra $(U(\mathbf{g})[[h]] \otimes U(\mathbf{g}))[[h]]$ with $F_{21} = F$ and $F = \mathbf{1} \otimes \mathbf{1} \bmod h$, and $[F, \Delta(x)] = 0$ for all $x \in A$, such that $A' = A_F$.

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